B.Sc. Part II DYNAMICS 4th Paper

Q. If the radial and transverse velocities ~~accelerations~~ of a particle are always proportional to each other, show that the path is an equiangular spiral.

If, in addition to above, the radial and transverse accelerations too are always proportional to each other, show that the velocity of the particle varies as some power of radius vector.

Soln Given that

radial velocity $\propto$ transverse velocity

$$\Rightarrow \frac{dr}{dt} \propto r\frac{d\theta}{dt}$$

$$\Rightarrow \frac{dr}{dt} = K r\frac{d\theta}{dt}, \quad K = \text{constant} \quad \text{— (1)}$$

$$\Rightarrow \frac{dr}{r} = K\,d\theta$$

integrating, we get

$$\Rightarrow \int \frac{dr}{r} = K\int d\theta$$

$$\Rightarrow \log r = K\theta + \log a$$

$\Rightarrow \log\frac{r}{a} = K\theta$

$\Rightarrow r = a e^{K\theta}$ which is an equiangular spiral.

Again, given that

radial acceleration $\propto$ transverse acceleration

$$\Rightarrow \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \propto \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$\Rightarrow \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \lambda \cdot \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$\Rightarrow \frac{d^2r}{dt^2} - r\cdot\frac{1}{K^2r^2}\left(\frac{dr}{dt}\right)^2 = \frac{\lambda}{r}\frac{d}{dt}\left(r^2\frac{1}{Kr}\frac{dr}{dt}\right)$$

$\left[\text{using (1) } \frac{d\theta}{dt} = \frac{1}{Kr}\frac{dr}{dt}\right]$

$$\Rightarrow \frac{d^2r}{dt^2} - \frac{1}{K^2r}\left(\frac{dr}{dt}\right)^2 = \frac{\lambda}{r}\frac{d}{dt}\left(\frac{r}{K}\frac{dr}{dt}\right)$$

$$\Rightarrow \frac{d^2r}{dt^2} - \frac{1}{K^2r}\left(\frac{dr}{dt}\right)^2 = \frac{\lambda}{r}\frac{1}{K}\frac{d}{dt}\left(r\frac{dr}{dt}\right)$$

$$\Rightarrow \frac{d^2r}{dt^2} - \frac{1}{K^2r}\left(\frac{dr}{dt}\right)^2 = \frac{\lambda}{Kr}\left[\left(\frac{dr}{dt}\right)^2 + r\frac{d^2r}{dt^2}\right]$$

$$\Rightarrow \frac{d^2r}{dt^2} - \frac{1}{K^2r}\left(\frac{dr}{dt}\right)^2 = \frac{\lambda}{Kr}\left(\frac{dr}{dt}\right)^2 + \frac{\lambda}{K}\frac{d^2r}{dt^2}$$

$$\Rightarrow \frac{d^2 r}{dt^2}\left(1-\frac{\lambda}{K}\right) = \frac{\lambda}{Kr}\left(\frac{dr}{dt}\right)^2 + \frac{1}{K^2 r}\left(\frac{dr}{dt}\right)^2$$

$$\Rightarrow \frac{d^2 r}{dt^2}\left(1-\frac{\lambda}{K}\right) = \left(\frac{dr}{dt}\right)^2 \cdot \frac{1}{r}\left(\frac{\lambda}{K}+\frac{1}{K^2}\right)$$

$$\Rightarrow \frac{d^2 r}{dt^2}\left(\frac{K-\lambda}{\cancel{K}}\right) = \frac{1}{r}\left(\frac{dr}{dt}\right)^2\left(\frac{\lambda K+1}{K^{\cancel{2}}}\right)$$

$$\Rightarrow \frac{d^2 r}{dt^2} = \frac{1}{r}\left(\frac{dr}{dt}\right)^2\left(\frac{\lambda K+1}{K(K-\lambda)}\right)$$

But $\frac{\lambda K+1}{K(K-\lambda)}$ = constant as λ and K are constants

let $\frac{\lambda K+1}{K(K-\lambda)} = c$ (suppose).

$$\Rightarrow \frac{d^2 r}{dt^2} = c \cdot \frac{1}{r}\left(\frac{dr}{dt}\right)^2$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{c}{r}\cdot\left(\frac{dr}{dt}\right)^2$$

$$\Rightarrow \frac{\frac{d}{dt}\left(\frac{dr}{dt}\right)}{\frac{dr}{dt}} = \frac{c}{r}\,\frac{dr}{dt}$$

Integrating w.r. to t, we get

$$\Rightarrow \log\left(\frac{dr}{dt}\right) = c\log r + \log c_1$$

$$\Rightarrow \frac{dr}{dt} = r^c \cdot c_1 \qquad \text{——— (2)}$$

Now, velocity $= \sqrt{(\text{radial velocity})^2 + (\text{transverse velocity})^2}$

$$\Rightarrow \text{velocity} = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2}$$

$$\Rightarrow \text{velocity} = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(\frac{1}{K}\frac{dr}{dt}\right)^2} \quad [\text{using (1)}]$$

$$\Rightarrow \text{velocity} = \frac{dr}{dt}\sqrt{1+\frac{1}{K^2}}$$

$$\Rightarrow \text{velocity} = \sqrt{1+\frac{1}{K^2}} \times \frac{dr}{dt}$$

$$= \sqrt{\frac{K^2+1}{K^2}} \times r^c \times c_1 \quad [\text{using (2)}]$$

But $c_1\sqrt{\frac{K^2+1}{K^2}} = \text{constant} = c_2 \text{ (say)}$

$$\Rightarrow \text{velocity} = c_2 \times r^c$$

$$\Rightarrow \text{velocity} \propto (\text{radius vector})^c$$

Proved